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# HUME AND BERKELEY ON THE PROOFS OF INFINITE DIVISIBILITY

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The philosophers of the seventeenth- and eighteenth-centuries were perplexed by and fascinated with the notion of infinite divisibility. On the one side, there seemed to be proofs, some of them geometrical demonstrations sound as one could imagine, showing, for example, that a line must be infinitely divisible. On the other side, the notion that a finite line could be made up of infinitely many parts struck many as being either unintelligible or absurd. Some philosophers were quick to put this apparent conflict to use. For Bayle the perplexities surrounding infinite divisibility evened the skeptical balance by providing arguments against the pretensions of *reason* on a par with the Cartesian assault on the *senses*.[1] Arnauld and Nicole enlisted the perplexities of infinite divisibility in the service of the mysteries of the Catholic faith, declaring that these proofs force us to confess "that there are some things which exist although we are not able to comprehend them"; then adding, "hence it is well for a man to weary himself with these subtilities, [sic.] in order to check his presumption, and to take away from him the boldness which would lead him to oppose his feeble intelligence to the truths which the Church proposes to him, under the pretext that he cannot understand them. . . ."[2]

Of course, *philosophical* arguments for and against infinite divisibility go back to the ancient world and in various places I shall discuss them in some detail, but I am chiefly concerned with the proposed *mathematical* proofs of infinite divisibility and the responses to them, specifically in the writings of Hume and Berkeley. In fact, most of their arguments are ineffective in refuting the mathematical proofs of infinite divisibility, sometimes

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[1]See Pierre Bayle's "Zeno of Elea," in his *Historical and Critical Dictionary*, translated with an introduction and notes, by Richard Popkin (Indianapolis, Ind.: Bobbs-Merrill, 1965).

[2]*The Port-Royal Logic,* translated with introduction, notes and appendix by Thomas Spencer Baynes,B.A., Second edition, enlarged (Edinburgh, Scotland: Sutherland and Know, 1851), pp. 307–308.

because they contain errors of reasoning and sometimes because they rely on philosophical commitments that are themselves dubious. In the end, however, we shall see that Berkeley did raise subtle and, I believe, original objections to these demonstrations that are independent of the problematic features of his general philosophical position, and of considerable interest in their own right.

## THE MATHEMATICAL PROOFS OF INFINITE DIVISIBILITY

Since the mathematical proofs of infinite divisibility were well known to the writers of the seventeenth- and eighteenth-centuries, they often alluded to them in a general way without spelling them out in detail. For this reason, it will be helpful to give a brief description of some of these proofs. One collection of them is found in *The Port-Royal Logic*, and it seems reasonable that writers interested in this topic would be familiar with this famous work. Another set of proofs of infinite divisibility is found in Isaac Barrow's *Lectiones Mathematicae*. Berkeley cites these lectures numerous times in his *Philosophical Commentaries*, and Bayle and Hume were familiar with them as well. In what follows, I shall primarily use the demonstrations found in *The Port-Royal Logic*, since they are elegantly stated and, presumably, widely known. I have taken one additional demonstration from Barrow's lectures because it is forceful, and because Berkeley responds to it in detail.

These mathematical demonstrations of infinite divisibility fall into two categories. The first employs arguments in the *reductio ad absurdum* form, that is, given the assumption that extension is *finitely* divisible, certain absurdities are supposed to follow, and it is therefore concluded that extension must be infinitely divisible.[3] For example, by a famous proof we know that the diagonal of a square is incommensurate with its sides, but if both the diagonal and the sides were composed of "a certain number of indivisible parts, one of these indivisible parts would be the common measure of these two lines, and, consequently, these two lines cannot be

[3]Bayle rejected this argument because he held that the notion of infinite divisibility leads to absurdities of its own. That is, he held that *divisibility* itself (whether finite or infinite) leads to absurdity. *Op cit.*, p. 366.

composed of a certain number of indivisible parts."[4] More elaborately:

> It is demonstrated, again, . . . that it is *impossible for a square number to be double of another square number*, while, however, it is very possible that an extended square may be double another extended square. Now, if these extended squares were composed of a certain number of ultimate parts, the large square would contain double the parts of the small one, and both being squares, there would be a square number double another square number, which is impossible.[5]

The second kind of mathematical argument in behalf of infinite divisibility employs constructions. *The Port-Royal Logic* contains one such argument which is intended to show that the infinite *extendability* of a line, which is countenanced in classical geometry, implies its infinite divisibility as well. The authors of The *Port-Royal Logic* ask us to imagine a ship sailing off on an infinitely broad flat sea. As the ship gets further away, its apparent size becomes smaller through, of course, infinitely many possible degrees, each corresponding to a possible division of a line.[6]

In another proof, this one found in Barrow's *Lectures,* a construction is used as the basis for a *reductio ad absurdum*:

> [I]f the Circumference of a Circle be supposed to consist of any Number of Points to every one of which *Radii* are drawn from the Center, it is very evident, that the Circumferences of more concentric Circles will consist of the same number of Points with the former, and consequently are equal to it, which is most absurd: Or otherwise these *Radii* do touch, meet, or intersect one another in some Place else than the Center.[7]

[4]*The Port-Royal Logic*, p. 306.

[5]Ibid., p. 306.

[6]Ibid., pp. 306–307.

[7]Isaac Barrow, *The Usefulness of Mathematical Learning Explained and Demonstrated: BEING MATHEMATICAL LECTURES READ IN THE PUBLIC SCHOOLS AT THE UNIVERSITY OF CAMBRIDGE*, Translated by the Reverend Mr. John Kirkby (London, England: Stephen Austin, 1734), p. 154. These lectures were presented at Cambridge University in the years 1664–66. They were first published, in Latin, in 1683.

A similar argument turns on the consideration that a set of lines parallel to one side of a square will put the adjoining sides of the square in one-to-one correlation with its diagonal. This suggests, though the argument demands elaboration, that the atomist is committed to holding that the

Finally, a more direct argument using a construction depends upon the claim that infinitely many angles can be drawn approximating the angle of intersection of a tangent to a circle.[8]

Arguments in response to these demonstrations move at two levels: (i) some ignore the details of these proofs and are content to point out supposed absurdities involved in the notion of infinite divisibility, whereas (ii) others attack the presented demonstrations themselves. In this essay I shall be primarily concerned with the second sort of response, but since it is often combined with more general philosophical arguments of the first kind, it will be helpful to note some of the general forms these arguments took. I shall only mention those that resurface in the writings of Berkeley and Hume.

## The Philosophical Arguments

Since antiquity, philosophers have presented arguments, some at a very high level of sophistication, for and against the possibility of infinite divisibility in various domains.[9] The seminal arguments in behalf of the infinite divisibility of magnitudes are found in Aristotle.[10] A favorite argument on the other side—Bayle, Berkeley, and Hume all rang changes on it[11]—is attributed to Epicurus by Lucretius:

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diagonal of a square is equal in length to the side of a square. Richard Sorabji has found this argument in the writings of the ninth-century Islamic philosopher Nazzam. See his *Time, Creation and the Continuum* (Ithaca, N.Y.: Cornell University Press, 1983), pp. 391–393.

[8]Hume cites this argument in the *Enquiry Concerning Human Understanding*, Introduction and Analytical Index by L.A. Selby-Bigge, third edition, with text revised and notes by P.H. Nidditch (Oxford, England: Oxford University Press, 1975), pp. 156–157.

[9]In recent years there has been a rise of interest in infinity, the continuum, and related topics as they were discussed in ancient and medieval times. Two works that provide an excellent entry into this field of philosophical scholarship are *Infinity and Continuity in Ancient and Medieval Thought*, edited by N. Kretzmann (Ithaca, N.Y.: Cornell University Press, 1982); and Richard Sorabji, *Time, Creation and the Continuum*, (n. 7 above).

[10]Aristotle's argument appears in various places, for example, *Categories* 4b22ff.; *Physics* V 3 and VI 1; *De gen. et corr.* I 2. These passages are conveniently collected in Appendix A in N. Kretzmann, editor, *Infinity and Continuity* (n. 9 above).

[11]For Bayle, *op cit.*, pp. 361–364.

> Moreover, if there be not a least thing, all the tiniest bodies will be composed of infinite parts, since indeed the half will always have a half, nor will anything set a limit. *What difference then will there be between the sum of things and the least things? There will be no difference,* for however completely the whole sum be infinite, yet things that are tiniest will be composed of infinite parts just the same.[12]

That is, as the italicized portion indicates, if extension were infinitely divisible, then everything would be the same size. The underlying argument seems to be this: if two things are composed of the same number of fundamental parts of the same size, then they must be equal in size; thus if every object were composed of infinitely many fundamental parts of the same size, then everything would be the same size. The argument is, of course, off the mark, since the believers in infinite divisibility deny that a line is composed of fundamental parts—a line is divisible all the way down without a stopping place.

Epicurus also argued, and Bayle, Berkeley, and Hume again followed him, that an extension composed of infinitely many finite extensions must constitute an infinite extension.[13] This argument isn't any good either. It is true that if we take a finite extension (however small) and repeat it *ad infinitum,* we will get an infinite extension. That, however, is quite beside the point, because the proof of infinite divisibility depends upon the possibility of constructing ever smaller finite extensions, as in the sequence [1/2, 1/4, 1/8, etc.] whose sum approaches, but does not exceed, 1.

Other philosophical arguments are reminiscent of Zeno's paradoxes.[14] Hume argued, in one place, that if extension were infinitely divisible, then no point could be specified as terminating a line, and, in another, that if time were infinitely divisible, it would be impossible for a finite time to elapse.[15] Again, these arguments

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[12]Titi Lucreti Cari, *De Rerum Natura,* edited and translated by Cyril Bailey, 3 vols. (Oxford, England: Clarendon Press, 1947) I, pp. 615–618, emphasis added.

[13]Epicurus, *Letter to Herodotus,* p. 57.

[14]For a superb discussion of Zeno's Paradoxes and the role they played in arguments against infinite divisibility, see Richard Sorabji, *Time, Creation and the Continuum* (n. 7 above), in particular Chapters 21 and 22.

[15]The first argument is found in *A Treatise of Human Nature,* Analytical Index by L. A. Selby-Bigge, second edition, with text revised and notes by P. H. Nidditch (Oxford, England: Oxford University Press, 1978), p. 44. The second argument is found in the *Enquiry,* p. 157.

seem to be based upon a misunderstanding of the concept of a limit, but I shall not discuss them further in this paper; not because their correct analysis is altogether easy, but because my interests lie elsewhere. I wish to examine the ways in which Hume and Berkeley responded directly to the mathematical proofs of infinite divisibility. After all, if these proofs are good proofs, then we might (as most mathematicians have done) reason in reverse fashion that there must be something wrong with the supposed derivations of Zeno-like paradoxes from the notion of the infinite divisibility of extension.

## HUME'S COUNTER-ARGUMENTS IN THE *TREATISE*

In arguing for his own position that space (and time) are the manner in which colored and tactile extensionless minima present themselves, Hume attacks what he takes to be all possible alternatives. *Pure mathematial points*, he claims, would be non-entities. *Physical points* (that is, *extended* minima) would be *in* space and hence divisible. Finally, he argues that extension cannot be *infinitely divisible*, for reasons we shall now examine.

## THE ARGUMENTS FROM MINIMAL SENSIBLES

Following Berkeley, and more remotely, Epicurus,[16] Hume attacks the doctrine of infinite divisibility by invoking the doctrine of minimal conceivables and sensibles. He first argues that at least our *ideas* of space and time are not infinitely divisible. If you take any object of the imagination, you will not be able to resolve it into infinitely many parts. Indeed, given an *idea* of a grain of sand, we will find that it is not even separable into twenty distinct ideas, "much less into a thousand, ten thousand, or an infinite number of ideas."[17] Hume makes a similar claim for impressions of sensation. Inspection shows that our capacity to discern distinct parts quickly reaches a limit, and we then encounter partless (and apparently extensionless) perceptual minima.[18]

[16]*Letter to Herodotus*, p. 58. For a discussion of Epicurus' appeals to minimal sensibles, see Sorabji, *op cit.*, pp. 371–372.
[17]*Treatise*, p. 27.
[18]Ibid., pp. 27–28.

So much for our *ideas* of space and time; how about space and time themselves? Are they infinitely divisible? Hume makes the transition from claims about our *ideas* of space and time to assertions about space and time *themselves* in the following remarkable passage which is a match for anything found in the writings of the rationalists.

> Wherever ideas are adequate representations of objects, the relations, contradictions and agreements of the ideas are all applicable to the objects; and this we may in general observe to be the foundation of all human knowledge. But our ideas are adequate representations of the most minute parts of extension; and thro' whatever divisions and subdivisions we may suppose these parts to be arriv'd at, they can never become inferior to some ideas, which we form. The plain consequence is, that whatever *appears* impossible and contradictory upon the comparison of ideas, must be *really* impossible and contradictory, without any further excuse or evasion.[19]

Given this principle, Hume restates one of the standard philosophical arguments against infinite divisibility as follows:

> I first take the least idea I can form of a part of extension, and being certain that there is nothing more minute than this idea, I conclude, that whatever I discover by its means must be a real quality of extension.[20]

Then, by starting with a particular finite quantity, Hume has no difficulty showing that their infinite repetition will generate an infinite quantity.

The difficulty with this argument is that the defenders of infinite divisibility do not hold that a line is composed of infinitely many finite parts of a fixed quantity; they hold that it is composed of infinitely many continuously diminishing finite quantities. Hume responds to this reply in a footnote:

> It has been objected to me, that infinite divisibility supposes only an infinite number of *proportional* not of *aliquot* parts, and that an infinite number of proportional parts does not form an infinite extension. But this distinction is entirely frivolous. Whether these parts be call'd *aliquot* or *proportional*, they cannot be inferior to those minute parts we

[19]Ibid., p. 29.
[20]Ibid.

> conceive; and therefore cannot form a less extension by their conjunction.[21]

Exactly why is it impossible for the parts of a line itself to be inferior to those minute parts we conceive? Hume's answer goes back to the claim that "our ideas are adequate representations of the most minute parts of extension," and "wherever ideas are adequate representations of objects, [they] are applicable to the objects." If this is the basis of Hume's argument, as it surely is, then two things are worth noting. (1) The resuscitation of this traditional argument against infinite divisibility is unnecessary since Hume could have argued directly that we have an adequate idea of extension as containing only finitely many minimal parts, and therefore extension itself has only finitely many minimal parts. (2) Hume certainly owes us a defense of the specific claim that we have an adequate idea of the ultimate parts of extension and also a defense of the general rationalist principle that adequate ideas of objects are *eo ipso* true of them. (If it is definitionally guaranteed that adequate ideas are true of their objects, then he owes us an account of adequacy.)

Hume's first use of the notion of perceptual minima as the basis for an argument against infinite divisibility of space and time rests upon the strong claim that our adequate ideas of extension guarantee certain things about the nature of extension itself. In this same context he offers, for the first time in his writings, a distinctive pattern of argument that he uses repeatedly throughout his philosophical career—we might call it the left tong of Hume's Fork. First he tells us:

> *That whatever the mind clearly conceives includes the idea of possible existence*, or in other words, *that nothing we imagine is absolutely impossible*.[22]

Now we know, or at least have been told, that we never imagine objects as having infinitely many parts, but rather we always imagine them as having finitely many parts. But, given the above *maxim of metaphysics*, as Hume calls it, he can argue in the following way: since we both imagine and perceive finite extensions as com-

[21]Ibid., 30n.
[22]Ibid., p. 32.

posed of only finitely many parts, that idea implies no contradiction; it is thus *possible* that space itself is only finitely divisible; and thus "all the arguments employ'd against the possibility of mathematical points [and in favor of infinite divisibility] are mere scholastick quibbles, and unworthy of our attention."[23]

The argument depends upon two theses: (i) we can (since, in fact, we always do) imagine extension as being composed of finitely many indivisible parts; and (ii) that which is imaginable is possible and thus cannot be demonstrated to be non-existent. I confess that I find the first principle baffling and hardly know what to say about it. I don't know how I would go about answering the question, "Does this line seem to you to be composed of finitely many or infinitely many parts?" It looks like something that will admit of a great many cuts, but will it admit of infinitely many cuts? I don't see how that question could be answered by inspection. Second, I find the argument from imaginability (or conceivability) unpersuasive since it ignores the possibility that we might misidentify or misdescribe the objects of our conception or imagination. But I shall not pursue that issue here. In any case, if Hume's argument were persuasive, we would know in advance that any proffered proof of infinite divisibility must be mistaken, and thus we would be under no constraint to examine these proofs before rejecting

[23]Ibid., p. 32. Writers who have compared Epicurus and Hume on minimal extensions sometimes ignore Hume's second argument from minimal sensibles. Sorabji, for example, claims that Epicurus' appeal to minimal sensibles was "intended to show how conceptually indivisible parts are *possible*, rather than to prove that they actually exist"; adding that "this makes Epicurus' argument differ at this point from those of Berkeley and Hume. Both of these thinkers argued from a minimum extension in our ideas to the conclusion that there must be, not merely that there can be, a minimum conceivable extension in reality" (*op cit.*, p. 72). Now in his first argument Hume does appeal to the existence of minimal perceivables to establish the *existence* of minimal extensions, but in this second argument they are invoked only to establish the *possibility* of minimal extensions. Hume then attempts to exploit the possibility of minimal extensions in a way *not* found in Epicurus, that is, to refute intended demonstrations of infinite divisibility.

Again, in the tenth chapter of his *Two Studies in the Greek Atomists* (Princeton, N.J.: Princeton University Press, 1967), David J. Furley elegantly details a wide range of similarities between the atomistic treatment of space and time found in Epicurus and in Hume. He does not, however, discuss Hume's second argument from perceptual minima and for this reason does not recognize at least one of the important differences between the treatment of indivisibles in Epicurus and Hume.

them. And that is what Hume says in calling them "mere scholastick quibbles . . . unworthy of our attention."[24] Yet Hume does attend directly to these 'demonstrations', I suppose because they have the power to persuade, and this should be neutralized.

## THE NON-DEMONSTRATIVE CHARACTER OF GEOMETRY

Hume's more direct attack upon these supposed proofs does not consist in finding errors in the derivations themselves; instead, it turns on a general claim concerning the nature of geometrical reasoning:

> I . . . maintain, that none of these demonstrations can have sufficient weight to establish such a principle, as this of infinite divisibility; and that because with regard to such minute objects, they are not properly demonstrations, being built on ideas, which are not exact, and maxims, which are not precisely true.[25]

According to Hume, we can never know, as geometers often claim they know, that two lines are exactly equal, for whether two lines are equal or not is a matter of observation, and when differences are minute, errors are possible. Similar remarks hold concerning the distinction between a right line and a curved line. At times it is beyond doubt that a line is curved rather than right, but close cases exist where we simply cannot tell the difference. From these observations Hume concludes that:

> It appears . . . that the ideas which are most essential to geometry, *viz.* those of equality and inequality, of a right line and a plain [sic.] surface, are far from being exact and determinate, according to our common method of conceiving them.[26]

I think that Hume wished to draw two conclusions from this claim: first, the general conclusion that geometry, despite a long tradition to the contrary, is an empirical discipline rather than a purely intellectual discipline; second, while admitting that geometry for the most part yields proofs that are quite beyond doubt, he further holds that it can lead to error when its reflections are

[24]Ibid., p. 32.
[25]Ibid., pp. 44–45.
[26]Ibid., pp. 50–51.

carried beyond the observational claims that serve as its basis. Geometry typically yields results that are beyond doubt because it proceeds from observations that are themselves beyond doubt. The situation is quite different with respect to the proofs of infinite divisibility. Here we are asked to make comparisons beyond our level of acuity, and, in the end, we arrive at results which, far from being supported by observation, actually contradict it.

There are, of course, a number of things wrong with Hume's argument. To begin with, in geometrical proofs, equalities are *stipulated* rather than discovered by observation. In geometry, lines are *set* equal to each other. Yet even if Hume is wrong in arguing that geometry is an empirical discipline on the grounds that it relies on observing the physical properties of diagrams, his *conclusion*, that geometry is an empirical discipline, may still be correct. He could have argued, as some modern empiricists have argued,[27] that when the non-logical terms of the propositions of geometry are given a physical interpretation, they become contingencies whose truth can only be established empirically.[28] If so, we get the result that Hume was trying to reach: the supposed *a priori* proofs of infinite divisibility do not show that *physical* space is infinitely divisible, since no purely demonstrative argument can establish the features of physical space. I shall return to this issue at the close of this essay.

## HINTS IN HUME'S *ENQUIRY*

When he came to write the *Enquiry Concerning Human Understanding*, Hume seems to have changed his mind about the non-demonstrative character of geometrical reasoning, for there he lists Geometry as one of the sciences derived wholly from Relations of Ideas.[29] Yet Hume's fascination with the problem of infinite divisibility is carried over to the *Enquiry*, where the discussion is curious and, in fact, not altogether forthcoming. In the body of the text he argues, in the style of Bayle, that the paradoxes gener-

[27]See, for example, C. G. Hempel's classic paper "Geometry and Empirical Science," *American Mathematical Monthly* 52 (1945).

[28]This is not the same as saying that they are empirical truths *because* they are contingencies.

[29]*Enquiry*, p. 25.

ated by the proofs of infinite divisibility should undermine the pretensions of reason. We find passages of the following kind:

> Reason here seems to be thrown into a kind of amazement and suspense, which, without the suggestions of any sceptic, gives her diffidence of herself, and of the ground on which she treads. She sees a full light, which illuminates certain places; but that light borders upon the most profound darkness. And between these she is so dazzled and confounded, that she scarcely can pronounce with certainty and assurance on any one object.[30]

But this brief alliance with Bayle is broken in a curious footnote appended to the close of the discussion. It begins with these words:

> It seems to me not impossible to avoid these absurdities and contradictions, if it be admitted, that there is no such thing as abstract or general ideas, properly speaking.[31]

Hume goes on to sketch, very briefly, his critique of what he takes to be the standard theory of abstract ideas and outlines his positive account of how general terms function. He then says that on this approach:

> [I]t follows that all the ideas of quantity, upon which mathematicians reason, are nothing but particular, and such as are suggested by the senses and imagination, and consequently, cannot be infinitely divisible.[32]

He ends coyly by telling us that "it is sufficient to have dropped this hint at present, without prosecuting it any further."[33] To my knowledge, Hume does not return to this topic in his published works.[34]

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[30]Ibid., p. 157.
[31]Ibid., 158n.
[32]Ibid.
[33]Ibid.
[34]In his correspondence, Hume mentions an apparently completed brief treatise on mathematical and physical subjects, but it was never published, and is now lost. Thus in a letter to Andrew Millar written on 12 June, 1755 he speaks of a work containing "Considerations previous to Geometry and Natural Philosophy." More than sixteen years later (on 25 January, 1772) he speaks of apparently the same work in a letter to William Strahan referring to it by the title *On Metaphisical Principles of Geom-*

In the *Treatise* Hume gives Berkeley highest marks for his critique of the received doctrine of abstract ideas, calling it "one of the greatest and most valuable discoveries that has been made of late years in the republic of letters."[35] In the passage just cited from the *Enquiry*, we again find Hume invoking Berkelean ideas concerning abstract ideas. Here, without saying so, Hume simply takes over Berkeley's extraordinarily sophisticated attack upon the standard proofs of infinite divisibility. This will become evident as we examine Berkeley's position on this matter.

## Berkeley's Critique of Infinite Divisibility

Berkeley discusses proofs for infinite divisibility in the *Principles of Human Knowledge*, (##123–132), in a set of queries (in particular, ##5–21) subjoined to the main text of *The Analyst* (a work *Addressed to an Infidel Mathematician*, in particular, Dr. Edmund Halley), and in his *Philosophical Commentaries*[36] where there is a large number of remarks bearing on infinite divisibility. We can notice first that two of Hume's main arguments against infinite divisibility appear in these passages. Like Hume, Berkeley invokes the Epicurean argument that an extension composed of infinitely many finite parts must be infinite in extension. Berkeley puts it this way: "when we say a line is infinitely divisible, we must mean a line which is infinitely great."[37]

Another of Hume's arguments, or at least its leading premise,

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*etry*. Both letters are found in *The Letters of David Hume*, edited by J. Y. T., Greig (Oxford, England: The Clarendon Press, 1932). Don Garrett pointed out these references to me.

[35]*Treatise*, p. 17. Remarkably, there are only two references to abstract ideas or abstracting in the *Enquiry Concerning Human Understanding* (the other is on pages 154–155). An entire section is dedicated to this topic in the *Treatise*, and Hume returns to the topic in a number of places later in that work.

[36]Quotations from Berkeley's writings come from *The Works of George Berkeley*, edited by A. A. Luce and T. E. Jessup, 9 vols. (London, England: Nelson, 1947–48). Citations are given to the sections of various works using the following abbreviations: *P* for *A Treatise Concerning the Principles of Human Knowledge*, *PC* for *Philosophical Commentaries*, and *A* for *The Analyst*. I have also profited from the citations and cross-references in A. A. Luce's editio diplomatica of the *Philosophical Commentaries*, (London, England: I. Nelson & Sons, 1955).

[37]*P*, #128.

also appears in Berkeley's writings: this is the argument from minimal perceptibles. Hume, as we saw, claimed that our *perceptions* of space and time are not infinitely divisible. Here's Berkeley:

> If . . . I cannot perceive innumerable parts in any finite extension that I consider, it is certain they are not contained in it: but it is evident, that I cannot distinguish innumerable parts in any particular line, surface, or solid, which I either perceive by sense, or figure to myself in my mind: wherefore I conclude they are not contained in it.[38]

Thus, for both Berkeley and Hume, it is a phenomenological fact that apprehended extensions are not infinitely divisible but, instead, are composed of finitely many *minimal* parts.

Given this shared premise concerning our idea of extension, Berkeley and Hume actually reason differently. Given his general position, Berkeley can conclude at once that extension *itself* is not infinitely divisible, for, as he says, "Nothing can be plainer to me than that the extensions I have in view are no other than my own ideas, and it is no less plain that I cannot resolve any one of my ideas into an infinite number of other ideas, that is, they are not infinitely divisible."[39] Since Hume maintains a distinction between extension and our idea of extension, he cannot argue in this way, and instead, must present arguments linking our ideas of extension to extension itself. We saw that he presented two: the stronger argument is intended to show that our adequate idea that extension is only finitely divisible guarantees that extension itself is only finitely divisible. The weaker argument is that there can be no de-

[38]*P*, #124. Berkeley had a remarkable response to those who argued that microscopes showed that objects are composed of unobservable parts. First, he recognized a threat from that direction, saying, "They who knew not [magnifying] Glasses had not so fair a pretense for Divisibility ad infinitum" (*PC*, #237). That is, those who have not viewed things under magnifying glasses would not be misled into thinking that such magnification could be carried on indefinitely, revealing more and more minute parts, thus showing that a line, for example, is infinitely divisible. Berkeley's rejoinder is that the line seen with the naked eye is not numerically identical with the line seen under the magnifying glass (see *PC*, #249). In *A New Theory of Vision*, he puts the matter this way: "A microscope brings us as it were into a new world: it presents us with a new scene of visible objects, quite different from what we behold with the naked eye" (#LXXXV).

[39]*P*, #124.

monstrative proofs of infinite divisibility since the *possibility* of only finite divisibility is established by the fact that we imagine and perceive extension as only finitely divisible; what is imaginable or perceivable is possible; and what is possible cannot be demonstrated not to exist. The following rough chart shows these three variations on the arguments from sensible minima:

| Both imagination and perception reveal minimal sensibles in extension. | | |
|---|---|---|
| 1. Our ideas of extension are adequate (true of) extension. | 2. Our ideas of extension are possibly true of extension. | 3. Extension has no existence outside the mind. |
| Therefore | | |
| Extension is not infinitely divisible. | There can be no proof that extension is infinitely divisible.[40] | Extension is not infinitely divisible. |

I confess that I find none of the arguments against infinite divisibility so far canvassed persuasive. The supposed derivations of contradictions from the notion that a finite extension has infinitely many finite parts rest (in both Berkeley and Hume) upon a misunderstanding of limits. The arguments from minimal sensibles, even if considered phenomenologically correct, rely, in each case, on further principles that are themselves dubious. The attack on the demonstrative character of mathematics rests, at least as Hume states it, on a misunderstanding of the character of geometrical reasoning. In fact, I think that this literature contains only one interestingly strong argument against the proofs for infinite divisibility. Hume apparently missed it, or failed to appreciate it, in his first foraging trip through Berkeley's writings, but he alludes to it, without specific acknowledgment, in the *Enquiry*. We will look at it next.

[40]In this argument Berkeley does not consider the possibility that God, at least, may comprehend extension as infinitely divisible. His response, I think, would be that such a speculation should be dismissed as unintelligible. We have no idea of something being infinitely divisible, and, where an idea is wanting, thought about it is impossible.

## Diagrams as Signs

It is a persistent theme of Berkeley's writing that the learned, though not often the vulgar, are constantly misled by the false doctrine of abstract ideas. Abstract ideas were used, among other things, to explain the function of general terms. Roughly, singular terms were said to stand for particular ideas, whereas general terms were said to stand for abstract ideas.[41] Berkeley objected to this on the grounds that there are no abstract ideas. Every idea of a triangle must be either scalene, right or obtuse, and no idea of a triangle can be more than one of these. In sum, for Berkeley, every idea must be a coherently determinate particular. How then do general terms achieve their generality? Berkeley's answer is that terms become general, not through their objects of reference, but through their modes of reference. General terms become general by indifferently referring to a range of particulars. Here I shall not comment on Berkeley's theory of general terms at large—there are troubles with it[42]—but concentrate on his application of these ideas to proofs involving geometrical diagrams.

Proofs from diagrams are puzzling. We establish something concerning a particular diagram on a piece of paper and then conclude that the same conclusion holds for all like figures. What justifies this apparently hasty generalization? The abstractionist might answer that the drawing of a particular triangle is merely a convenient stand-in for our abstract idea of a triangle. Rejecting abstract ideas, Berkeley gives a wholly different account of the generality of proofs that rely on diagrams.

> [T]he particular lines and figures included in the diagram, are supposed to stand for innumerable objects of different sizes: or in other words, the geometer considers them abstracting from their magnitude: which does not imply that he forms an abstract idea, but only that he cares not what the particular magnitude is, whether great or small, but looks on that as a thing indifferent to the demonstration. . . .[43]

[41]Locke, who is the specific target of Berkeley's attack on abstract ideas, is not treated altogether fairly by him.

[42]In particular, in its attack on abstract ideas, it relies heavily on an imagist account of concepts.

[43]*P*, #126. Berkeley first makes this same point in ##15 and 16 of the Introduction.

Thus it is the way in which the diagram is employed, not its reference to some special kind of object, that gives it its generality. We could also say that a diagram is not a flawed attempt to picture an abstract idea, but instead a *sign* that can be successfully used to refer to indefinitely many particulars. Berkeley puts it just this way in Query #6 in the *Analyst*:

> Whether the diagrams in geometrical demonstrations are not to be considered as signs, of all possible finite figures, of all sensible and imaginable extensions or magnitudes of the same kind?

The expected answer to this question is yes.

But those who are under the influence of the theory of abstract ideas are liable to misunderstand the manner in which diagrams are employed.[44] Indeed, the tendency toward misunderstanding seems entirely *natural*, for:

> a line in the scheme, but an inch long, must be spoken of, as though it contained ten thousand parts, since it is regarded not in it self, but as it is universal; and it is universal only in its signification, whereby it represents innumerable lines greater than it self, in which may be distinguished ten thousand parts or more, though there may not be above an inch in it.

Then the key passage:

> After this manner the properties of the lines signified are (by a very usual figure) transferred to the sign, and thence through mistake thought to appertain to it considered in its own nature.[45]

Berkeley's point here is remarkably subtle. He is not simply saying, as Wittgenstein would,[46] that a misunderstanding of the

[44]See Query 7 in the *Analyst*.

[45]*P*, #126. Query 17 in the *Analyst* makes the same point this way: "Whether the considering geometrical diagrams absolutely or in themselves, rather than as representatives of all assignable magnitudes or figures of the same kind, be not a principal cause of the supposing finite extension infinitely divisible; and of all the difficulties and absurdities consequent thereupon?"

[46]The remarkable similarity between Berkeley's and Wittgenstein's treatment of mathematics is obvious. ##119–122 of the *Principles*, which concerns arithmetic, is similarly very close to Wittgenstein.

way in which a sign functions can lead us to posit curious entities, for example, the abstract ideas of the conceptualist or the abstract particulars of the Platonic realist. He is further indicating that such misunderstandings can foster absurd views about objects right before our eyes: that, for example, we can be led to think that a one-inch line drawn on a page must be composed of infinitely many (mostly indiscernible) parts.

But even if Berkeley's arguments are remarkably subtle, are they sound? There are a number of things worth saying on this score. First, unlike his other arguments intended to show the absurdity of the notion of infinite divisibility, this argument does not depend upon misunderstandings of the notion of a limit. Second, unlike the argument from minimal sensibles, it does not rely on any of the dubious features of his general subjective idealist position. *Indeed, his argument can be stated, and for the most part is stated, in a way that is independent of his general immaterialist philosophical position.* Finally, unlike his other arguments, it is not intended to show that extension is only finitely divisible. Its point is wholly negative: it is intended to show that certain so-called proofs of infinite divisibility are not really proofs at all.

But isn't Berkeley wrong about this? Aren't the proofs from *The Port-Royal Logic* that we began with as forceful as any that can be imagined? Don't these proofs rest on their own bottoms? Consider the Port-Royal proof about the ship moving away over an infinitely flat plane. Doesn't that show that a line admits of infinitely many degrees of diminution and hence must be infinitely divisible? To answer this, it is important to see that this argument is not a proof or demonstration in the sense that the standard derivation of, say, the Pythagorean Theorem is a proof or demonstration. In that case the formula for the Pythagorean Theorem is the last step in the derivation; it occurs as a line in a proof. Here, instead, we are presented with a vivid example which tempts us to view things in a certain way. The fact that, presented with this picture, we are strongly inclined to accept the conclusion that a line is infinitely divisible does not convert the reasoning into a geometrical derivation. Furthermore, if Berkeley is right, he has explained, and explained away, this temptation.[47]

[47]Although this is not the place to go into detail, it is worth noting that Berkeley takes essentially the same nominalistic approach to arithmetic.

## Countering The Indirect Proofs of Infinite Divisibility

We saw at the beginning that two sorts of mathematical proofs were available for the infiinite divisibility of lines: some of these arguments involved geometrical constructions of the kind that we have just been examining; others involved indirect proofs that depended upon showing the absurdity involved in the supposition of only *finite* divisibility. We can begin with the first argument mentioned, that is, if lines were only finitely divisible, then, *per impossibile*, the diagonal of a square would be commensurate with its sides.[48]

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Here too he argues that philosophers have misunderstood the way mathematical signs, this time numerals, function.

> In *arithmetic*, . . . we regard not *things* but the *signs*, which nevertheless are not regarded for their own sake, but because they direct us how to act with relation to things, and dispose rightly of them. . . . [T]hose things which pass for abstract truths and theorems concerning numbers, are, in reality, conversant about no object distinct from particular numeral things, except only names and characters which originally came to be considered, on no other account but their being *signs*, or capable to represent aptly, whatever particular things men had need to compute. Whence it follows, that to study them for their own sake would be just as wise, and to as good a purpose, as if a man, neglecting the true use or original intention and subserviency of language, should spend his time in impertinent criticisms upon words, or reasonings and controversies purely verbal (*P*, #122).

I think that this passage has a deep point hidden under a more superficial point. The more superficial point is that mathematicians waste their time when they attempt to establish truths of pure mathematics. The proof, for example, that there is no greatest prime may be both elegant and ingenious, but will serve no useful purpose in "directing us how to act with relations to things and of disposing rightly of them." The deeper point is that arithmetic has no subject matter of its own in the sense of a system of *entities* somehow beyond particular signs and the things enumerated by these signs. Just as with geometry, the assumption that such special mathematical entities must exist is the result of misunderstanding the way in which certain mathematical signs function, a misunderstanding seconded by the faulty doctrine of abstract ideas.

[48]The second indirect proof in *The Port-Royal Logic*, though different in detail, turns on the same basic consideration, that is, that the square root of 2 is irrational. If a square number, that is, a number with an integral square root, could be the double of another square number, then the following equation would hold: $x^2 = 2y^2$. Solving for x, we get that $x = y * \sqrt{2}$. Since the square root of 2 is irrational, x can have no integral solutions. But, and this begins the *reductio*, if extension were only finitely divisible, then every area would be composed of only finitely many minimal

In order to examine possible responses, let me fill out this argument in detail. If a line were composed of finitely many *equal* minimum parts, then, so the argument goes, the diagonal would have some finite number of such parts *d*, and the sides would have some finite number of such parts *s*. The ratio between them would be the rational fraction *d/s*. We know from the Pythagorean Theorem that the ratio of the diagonal of a square to its sides is the square root of 2 to 1. We also know from a famous proof that the square root of 2 is irrational, that is, it cannot be expressed as a rational fraction. Thus the doctrine of finite divisibility seems to be in conflict with at least one central theorem of classical mathematics.

As far as I know, Berkeley never discusses this argument for infinite divisibility in the writings he published, but in the *Philosophical Commentaries* he discusses it explicitly and at length. In an early entry he asks himself whether the incommensurability of the diagonal with the side is consistent with his principle. Two hundred thirty-six entries later he declares flatly:

> The Diagonal is commensurable with the Side.[49]

But if Berkeley is going to maintain this thesis, something in the classical line of reasoning sketched above must be rejected. At one point he attacks the Pythagorean Theorem, saying:

> One square cannot be double of another. Hence the Pythagoric Theorem is false.[50]

This, in fact, is a straightforward consequence of Berkeley's position, but it may help to begin with an easier case. For Berkeley, any diagram that is actually drawn must have commensurable components.

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areas and there would, *per impossibile,* be an integral solution to this equation. (Incidentally, this argument fails because it turns on the assumption that there must be a square corresponding to every assignable area. A believer in minimal extensions would deny this—a point that will become clear when we examine the simpler argument concerning the incommensurability of the diagonal of a square with its sides.)

[49]*PC*, #264.

[50]Ibid., #500.

> Diagonal of particular square commensurable with its side they both containing a certain number of M: V: [that is, minimal visibles].[51]

This is a fact that will hold quite generally for all figures; for example, there must be a rational ratio between the diameter and the circumference of a circle. From this it follows that actual figures never exemplify a true square with its diagonal perfectly inscribed. If the square is perfect, then the actual diagonal must either fall short of one of the corners of the square or overshoot it. If the diagonal does not overshoot or undershoot the corners, then the square cannot be perfect. These results may seem outrageous, but they are, after all, natural consequences of the doctrine of minimal sensibles. And now we can see why the Pythagorean Theorem must be false. Just as there cannot be true squares with exactly inscribed diagonals, there cannot be squares corresponding to every possible area. For example, there could not be a square composed of 137 minimal visibles, since 137 does not have an integral square root.[52]

It is clear from a number of passages in his *Philosophical Commentaries* that Berkeley was not shy about embracing such results. There, we find him making remarks of the following kind:

> It seems that all lines can't be bisected in 2 equall parts. Mem: to examine how the Geometers prove the contrary.[53]

If a line were composed of an odd number of MV's, then it would not be possible to divide it into two equal parts. Or again:

> It seems all Circles are not similar figures there not being the same proportion betwixt all circumferences & their diameters.[54]

That is, those things that we commonly, and, for Berkeley, *rightly*, call circles will not all be strictly congruent because of the constructive contraints put on figures by minimal visibles. Anyway, Berkeley's fundamental argument is that these proofs that appeal to incommensurability cannot be used to prove that extension is

[51]Ibid., #258.

[52]Actually, there is a problem here that Berkeley was aware of but does not face up to: what are the *shapes* of minimal visibles? Could they come in a variety of shapes? If they are square, for example, then squares of various sizes, but not all sizes, could be built up out of them, but, of course, no true triangles or circles could be constructed.

[53]Ibid., #276.

[54]Ibid., #340.

infinitely divisible, since the possibility of constructing lines, areas, etc., of any assignable size and shape presupposes that infinite divisibility is possible and thus cannot be used to prove it.

> Mem. To Enquire most diligently Concerning the Incommensurability of Diagonal & side. whether it Does not go on the supposition of unit being divisible ad infinitum, i.e. of the extended thing spoken of being divisible ad infinitum (unit being nothing also V. Barrow Lect. Geom:). & so the infinite [divisibility] deduc'd therefrom is a petitio principii.[55]

## PHYSICAL SPACE VS. PURE SPACE

Still, isn't it just obvious that a line is infinitely divisible, for if we cut it in half, there will always be something left over to cut in half again?[56] But is that even true? With a pencilled line we finally get down to gaps between the pieces of graphite, and whatever method is used to produce the line, we will finally arrive at atomic gaps where there is nothing available to divide. And this is precisely how Berkeley can deal with Barrow's concentric circle argument for infinite divisibility.[57] Given two concentric circles, Barrow tells us, in effect, that we can put the points on the outer circle into one-to-one correlation with the points on the inner circle by drawing lines from the outer circle through the inner circle to the common center. This, he maintains, would not be possible if these circumferences contained only finitely many points. For Berkeley this is false, and just obviously false, for when we actually attempt to perform this feat, with a pencil however sharp, we discover that the lines merge together before they reach the center point.

To this, the defender of infinite divisibility will reply that it is not the *physical* lines that are here in question, but the pure widthless *mathematical* lines. Berkeley, of course, will have none of this:

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[55]Ibid., #263. The text actually reads "the infinite *in*divisibility deduc'd therefrom is a petitio principii." This is obviously a slip, since the text makes no sense read that way.

[56]The *Port-Royal* logicians put it this way: ". . . there is nothing more clear than this principle, *that two non-extensions cannot form an extension,* and that an extended whole has parts. Now, taking two of these parts, which we assume to be indivisible, I ask, whether these have extension, or whether they have not? If they have, they are therefore divisible, and have many parts; if they have not, they are two negations of extension, and thus it is impossible for them to constitute an extension." *Op cit.*, 306.

[57]Berkeley cites this proof at *PC*, #315.

> The Mathematicians think there are insensible lines, about these they harangue, these cut in a point, at all angles these are divisible ad infinitum. We Irish men can conceive no such lines.[58]

## CONCLUSION

I don't want to take sides on this debate, but it should be clear that those who maintain that segments of extension are infinitely divisible (whether they are exemplified or not) are thereby committing themselves to the existence of Platonic abstract particulars. Or, to take sides a bit, it is not implausible to suppose that the question whether *physical space* is infinitely divisible is an empirical question, albeit a high-level, theory-laden empirical question. If in response it is argued that *pure space*, at least, is infinitely divisible, then we are owed an explanation of the nature of pure space and a further explanation of its relation to physical space. Now the burden of embarrassment shifts from Berkeley's shoulders to those of his opponents.

My own inclination is to think the question *Is extension infinitely divisible*? has a clear sense only when it is interpreted as meaning *Is physical space infinitely divisible*? and that question in turn expands into the further question: *What kind of mathematics is needed to generate the best theory of physical reality*? So far, the answer seems to be that the real number system forms part of the system that provides the most adequate interpretation of physical reality. That, of course, carries the implication that physical space is, after all, infinitely divisible. Still, this thesis about physical space cannot be established by the kind of *a priori* arguments that Berkeley rejected. My tentative conclusion is that Berkeley may well have been wrong in denying that physical space is infinitely divisible,[59] but he was surely right in denying that this thesis can be established using the traditional proofs for infinite divisibility of the kind we have examined.[60]

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[58] *PC*, #393.

[59] In denying the existence (and intelligibility) of what philosophers call matter (or corporeal substance), Berkeley was not, of course, denying the existence of physical space. Berkeley, along with others, thought, for example, that Dublin was some distance from London.

[60] My colleagues Timothy Duggan and Richard Kremer have made a great many important suggestions concerning this paper. I am also indebted to the editors of this journal for their helpful suggestions.